

EXTENSIONS OF THE BENSON-SOLOMON FUSION SYSTEMS

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To Dave Benson on the occasion of his second 60th birthday

ABSTRACT. The Benson-Solomon systems comprise the only known family of simple saturated fusion systems at the prime two that do not arise as the fusion system of any finite group. We determine the automorphism groups and the possible almost simple extensions of these systems and of their centric linking systems.

1. INTRODUCTION

Given a group \hat{G} with normal subgroup G such that $C_{\hat{G}}(G) \leq G$, the structure of \hat{G} is controlled by G and $\text{Aut}(G)$. On the other hand, if $\hat{\mathcal{F}}$ is a saturated fusion system with normal subsystem \mathcal{F} such that $C_{\hat{\mathcal{F}}}(\mathcal{F}) \leq \mathcal{F}$, it is in general more difficult to describe the possibilities for $\hat{\mathcal{F}}$ given \mathcal{F} . For example, if $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of the finite group G , then it is often the case that some extensions of \mathcal{F} do not arise as extensions of G , but rather as extensions of a different finite group with the same fusion system. It is also possible that there are extensions of $\mathcal{F}_S(G)$ that are *exotic* in the sense that they are induced by no finite group, but to our knowledge, no example of this is currently known.

It has become clear that working with the fusion system alone is not on-its-face enough for describing extensions [Oli10, AOV12, Oli16, BMO16]; one should instead work with an associated linking system. In particular, when $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$ for some finite group G , there is a natural map $\kappa_G: \text{Out}(G) \rightarrow \text{Out}(\mathcal{L})$, and whether the extensions of \mathcal{F} all come from finite groups has been shown to be related to asking whether $\kappa_{G'}$ is split surjective for some other finite group G' with the same Sylow subgroup and with $\mathcal{F} = \mathcal{F}_S(G')$ [AOV12, Oli16]. While this machinery is not directly applicable to the problem of determining extensions of exotic fusion systems, there are other tools for use, such as [Oli10].

There is one known family of simple exotic fusion systems at the prime 2, the Benson-Solomon systems. They were first predicted by Dave Benson [Ben98] to exist as finite versions of a 2-local compact group associated to the 2-compact group $DI(4)$ of Dwyer and Wilkerson [DW93]. They were later constructed by Levi and Oliver [LO02] and Aschbacher and Chermak [AC10]. The purpose of this paper is determine the automorphism groups of the Benson-Solomon fusion and centric linking systems, and use that information to determine the fusion systems having one of these as its generalized Fitting subsystem. This information is needed within certain portions of Aschbacher's program to classify simple fusion systems of component type at the prime 2. In particular, it is presumably needed within an involution centralizer problem for

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these systems. Some of the work on automorphisms of these systems appears in the standard references [LO02, LO05, AC10], and part of our aim is to complete the picture.

The plan for this paper is as follows. All maps are written on the left. In Section 2, we recall the various automorphism groups of fusion and linking systems and the maps between them, following [AOV12]. In Section 3, we look at automorphisms of the fusion and linking systems of $\text{Spin}_7(q)$ and of the Benson-Solomon systems. We show in Theorem 3.10 that the outer automorphism group of the latter is a cyclic group of field automorphisms of 2-power order. Finally, we show in Theorem 4.3 that the systems having a Benson-Solomon generalized Fitting subsystem are uniquely determined by the outer automorphisms they induce on the fusion system, and that all such extensions are split.

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2. AUTOMORPHISMS OF FUSION AND LINKING SYSTEMS

We refer to [AKO11] for the definition of a saturated fusion system, and also for the definition of a centric subgroup of a fusion system. Let \mathcal{F} be a saturated fusion system over the finite p -group S , and write \mathcal{F}^c for the collection of \mathcal{F} -centric subgroups. Whenever g is an element of a finite group, we write c_g for the conjugation homomorphism $x \mapsto {}^g x = gxg^{-1}$ and its restrictions.

2.1. Background on linking systems. Whenever Δ is an overgroup-closed, \mathcal{F} -invariant collection of subgroups of S , we have the transporter category $\mathcal{T}_\Delta(S)$ with those objects. This is the full subcategory of the transporter category $\mathcal{T}_S(S)$ where the objects are subgroups of S , and morphisms are the transporter sets: $N_S(P, Q) = \{s \in S \mid sPs^{-1} \leq Q\}$ with composition given by multiplication in S .

A *linking system* associated to \mathcal{F} is a nonempty category \mathcal{L} with object set Δ , together with functors

$$(2.1) \quad \mathcal{T}_\Delta(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}.$$

The functor δ is the identity on objects and injective on morphisms, while π is the inclusion on objects and surjective on morphisms. Write $\delta_{P,Q}$ for the corresponding injection $N_S(P, Q) \rightarrow \text{Mor}_{\mathcal{L}}(P, Q)$ on morphisms, write δ_P for $\delta_{P,P}$, and use similar notation for π .

The category and its structural functors are subject to several axioms which may be found in [AKO11, Definition II.4.1]. In particular, Axiom (B) states that for all objects P and Q in \mathcal{L} and each $g \in N_S(P, Q)$, we have $\pi_{P,Q}(\delta_{P,Q}(g)) = c_g \in \text{Hom}_{\mathcal{F}}(P, Q)$. A *centric linking system* is a linking system with $\Delta = \mathcal{F}^c$. Given a finite group G with Sylow p -subgroup S , the canonical centric linking system for G is the category $\mathcal{L}_S^c(G)$ with objects the p -centric subgroups $P \leq S$ (namely those P whose centralizer satisfies $C_G(P) = Z(P) \times O_{p'}(C_G(P))$), and with morphisms the orbits of the transporter set $N_G(P, Q) = \{g \in G \mid gPg^{-1} \leq Q\}$ under the right action of $O_{p'}(C_G(P))$.

2.1.1. Distinguished subgroups and inclusion morphisms. The subgroups $\delta_P(P) \leq \text{Aut}_{\mathcal{L}}(P)$ are called *distinguished subgroups*. When $P \leq Q$, the morphism $\iota_{P,Q} := \delta_{P,Q}(1) \in \text{Mor}_{\mathcal{L}}(P, Q)$ is the *inclusion* of P into Q .

2.1.2. Axiom (C) for a linking system. We will make use of Axiom (C) for a linking system, which says that for each morphism $\varphi \in \text{Mor}_{\mathcal{L}}(P, Q)$ and element $g \in N_S(P)$, the following identity holds between morphisms in $\text{Mor}_{\mathcal{L}}(P, Q)$:

$$\varphi \circ \delta_P(g) = \delta_Q(\pi(\varphi)(g)) \circ \varphi.$$

2.1.3. Restrictions in linking systems. For each morphism $\psi \in \text{Mor}_{\mathcal{L}}(P, Q)$, and each $P_0, Q_0 \in \text{Ob}(\mathcal{L})$ such that $P_0 \leq P$, $Q_0 \leq Q$, and $\pi(\psi)(P_0) \leq Q_0$, there is a unique morphism $\psi|_{P_0, Q_0} \in \text{Mor}_{\mathcal{L}}(P_0, Q_0)$ (the *restriction* of ψ) such that $\psi \circ \iota_{P_0, P} = \iota_{Q_0, Q} \circ \psi|_{P_0, Q_0}$; see [Oli10, Proposition 4(b)] or [AKO11, Proposition 4.3].

Note that in case $\psi = \delta_{P, Q}(s)$ for some $s \in N_S(P, Q)$, it can be seen from Axioms (B) and (C) that $\psi|_{P_0, Q_0} = \delta_{P_0, Q_0}(s)$.

2.2. Background on automorphisms.

2.2.1. Automorphisms of fusion systems. An automorphism of \mathcal{F} is, by definition, determined by its effect on S : define $\text{Aut}(\mathcal{F})$ to be the subgroup of $\text{Aut}(S)$ consisting of those automorphisms α which *preserve fusion* in \mathcal{F} in the sense that the homomorphism given by $\alpha(P) \xrightarrow{\alpha\varphi\alpha^{-1}} \alpha(Q)$ is in \mathcal{F} for each morphism $P \xrightarrow{\varphi} Q$ in \mathcal{F} . The automorphisms $\text{Aut}_{\mathcal{F}}(S)$ of S in \mathcal{F} thus form a normal subgroup of $\text{Aut}(\mathcal{F})$, and the quotient $\text{Aut}(\mathcal{F})/\text{Aut}_{\mathcal{F}}(S)$ is denoted by $\text{Out}(\mathcal{F})$.

2.2.2. Automorphisms of linking systems. A self-equivalence of \mathcal{L} is said to be *isotypical* if it sends distinguished subgroups to distinguished subgroups, i.e. $\alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$ for each object P . It sends inclusions to inclusions provided $\alpha(\iota_{P, Q}) = \iota_{\alpha(P), \alpha(Q)}$ whenever $P \leq Q$. The monoid $\text{Aut}(\mathcal{L})$ of isotypical self-equivalences that send inclusions to inclusions is in fact a group of automorphisms of the category \mathcal{L} , and this has been shown to be the most appropriate group of automorphisms to consider. Note that $\text{Aut}(\mathcal{L})$ has been denoted by $\text{Aut}_{\text{typ}}^I(\mathcal{L})$ in [AKO11, AOV12] and elsewhere. When $\alpha \in \text{Aut}(\mathcal{L})$ and P is an object with $\alpha(P) = P$, we denote by α_P the automorphism of $\text{Aut}_{\mathcal{L}}(P)$ induced by α .

The group $\text{Aut}_{\mathcal{L}}(S)$ acts by conjugation on \mathcal{L} in the following way: given $\gamma \in \text{Aut}_{\mathcal{L}}(S)$, consider the functor $c_{\gamma} \in \text{Aut}(\mathcal{L})$ which is $c_{\gamma}(P) = \pi(\gamma)(P)$ on objects, and which sends a morphism $P \xrightarrow{\varphi} Q$ in \mathcal{L} to the morphism $\gamma\varphi\gamma^{-1}$ from $c_{\gamma}(P)$ to $c_{\gamma}(Q)$ after replacing γ and γ^{-1} by the appropriate restrictions (introduced in §2.1.3). Note that when $\gamma = \delta_S(s)$ for some $s \in S$, then $c_{\gamma}(P)$ is conjugation by s on objects, and $c_{\gamma}(\varphi) = \delta_{Q, sQ}(s) \circ \varphi \circ \delta_{sP, P}(s^{-1})$ for each morphism $\varphi \in \text{Mor}_{\mathcal{L}}(P, Q)$ by the remark on distinguished morphisms in §2.1.3. In particular, when $\mathcal{L} = \mathcal{L}_S^c(G)$ for some finite group G , c_{γ} is truly just conjugation by s on morphisms.

The image of $\text{Aut}_{\mathcal{L}}(S)$ under the map $\gamma \mapsto c_{\gamma}$ is seen to be a normal subgroup of $\text{Aut}(\mathcal{L})$; the outer automorphism group of \mathcal{L} is

$$\text{Out}(\mathcal{L}) := \text{Aut}(\mathcal{L})/\{c_{\gamma} \mid \gamma \in \text{Aut}_{\mathcal{L}}(S)\};$$

we refer to Lemma 1.14(a) and the surrounding discussion in [AOV12] for more details. This group is denoted by $\text{Out}_{\text{typ}}(\mathcal{L})$ in [AKO11, AOV12] and elsewhere.

2.2.3. From linking system automorphisms to fusion system automorphisms. There is a group homomorphism

$$(2.2) \quad \tilde{\mu}: \text{Aut}(\mathcal{L}) \longrightarrow \text{Aut}(\mathcal{F}),$$

given by restriction to $S \cong \delta_S(S) \leq \text{Aut}_{\mathcal{L}}(S)$; see [Oli10, Proposition 6]. The map $\tilde{\mu}$ induces a homomorphism on quotient groups

$$\mu: \text{Out}(\mathcal{L}) \longrightarrow \text{Out}(\mathcal{F}).$$

We write $\mu_{\mathcal{L}}$ (or μ_G when $\mathcal{L} = \mathcal{L}_S^c(G)$) whenever we wish to make clear which linking system we are working with; similar remarks hold for $\tilde{\mu}$. As shown in [AKO11, Proposition II.5.12], $\ker(\mu)$ has an interesting cohomological interpretation as the first cohomology group of the center functor

$Z_{\mathcal{F}}$ on the orbit category of \mathcal{F} -centric subgroups, and $\ker(\tilde{\mu})$ is correspondingly a certain group of normalized 1-cocycles for this functor.

2.2.4. From group automorphisms to fusion system and linking system automorphisms. We also need to compare automorphisms of groups with the automorphisms of their fusion and linking systems. If G is a finite group with Sylow p -subgroup S , then each outer automorphism of G is represented by an automorphism that fixes S . This is a consequence of the transitive action of G on its Sylow subgroups. More precisely, there is an exact sequence:

$$1 \rightarrow Z(G) \xrightarrow{\text{incl}} N_G(S) \xrightarrow{g \mapsto c_g} \text{Aut}(G, S) \rightarrow \text{Out}(G) \rightarrow 1.$$

where $\text{Aut}(G, S)$ is the subgroup of $\text{Aut}(G)$ consisting of those automorphisms that leave S invariant.

For each pair of p -centric subgroups $P, Q \leq S$ and each $\alpha \in \text{Aut}(G, S)$, α induces an isomorphism $O_{p'}(C_G(P)) \rightarrow O_{p'}(C_G(\alpha(P)))$ and a bijection $N_G(P, Q) \rightarrow N_G(\alpha(P), \alpha(Q))$. Thus, there is a group homomorphism

$$\tilde{\kappa}_G: \text{Aut}(G, S) \rightarrow \text{Aut}(\mathcal{L}_S^c(G))$$

which sends $\alpha \in \text{Aut}(G, S)$ to the functor which is α on objects, and also α on morphisms in the way just mentioned. This map sends the image of $N_G(S)$ to $\{c_\gamma \mid \gamma \in \text{Aut}_{\mathcal{L}_S^c(G)}(S)\}$, and so induces a homomorphism

$$\kappa_G: \text{Out}(G) \rightarrow \text{Out}(\mathcal{L}_S^c(G))$$

on outer automorphism groups.

It is straightforward to check that the restriction to S of any member of $\text{Aut}(G, S)$ is an automorphism of the fusion system $\mathcal{F}_S(G)$. Indeed, for every $\alpha \in \text{Aut}(G, S)$, the automorphism $\alpha|_S$ of $\mathcal{F}_S(G)$ is just the image of α under $\tilde{\mu}_G \circ \tilde{\kappa}_G$.

2.2.5. Summary. What we will need in our proofs is summarized in the following commutative diagram, which is an augmented and updated version of the one found in [AKO11, p.186].

$$(2.3) \quad \begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ Z(\mathcal{F}) & \xrightarrow{\text{incl}} & Z(S) & \longrightarrow & \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) & \longrightarrow & \varprojlim^1(\mathcal{Z}_{\mathcal{F}}) \longrightarrow 1 \\ \parallel & & \downarrow \delta_S & & \downarrow \tilde{\lambda} & & \downarrow \lambda \\ Z(\mathcal{F}) & \longrightarrow & \text{Aut}_{\mathcal{L}}(S) & \longrightarrow & \text{Aut}(\mathcal{L}) & \longrightarrow & \text{Out}(\mathcal{L}) \longrightarrow 1 \\ & & \downarrow \pi_S & & \downarrow \tilde{\mu} & & \downarrow \mu \\ 1 & \longrightarrow & \text{Aut}_{\mathcal{F}}(S) & \longrightarrow & \text{Aut}(\mathcal{F}) & \longrightarrow & \text{Out}(\mathcal{F}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

All sequences in this diagram are exact. Most of this either is shown in the proof of [AKO11, Proposition II.5.12], or follows from the above definitions. The first and second rows are exact by this reference, except that the diagram was not augmented by the maps out of $Z(\mathcal{F})$ (the center of \mathcal{F}); exactness at $Z(S)$ and $\text{Aut}_{\mathcal{L}}(S)$ is shown by following the proof there. Given [AKO11, Proposition II.5.12], exactness of the last column is equivalent to the uniqueness of centric linking

systems, a result of Chermak. In all the cases needed in this article, it follows from [LO02, Lemma 3.2]. The second-to-last column is then exact by a diagram chase akin to that in a 5-lemma for groups.

3. AUTOMORPHISMS

The isomorphism type of the fusion systems of the Benson-Solomon systems $\mathcal{F}_{\text{Sol}}(q)$, as q ranges over odd prime powers, is dependent only on the 2-share of $q^2 - 1$ by [COS08, Theorem B]. Since the centralizer of the center of the Sylow group is the fusion system of $\text{Spin}_7(q)$, the same holds also for the fusion systems of these groups. For this reason, and because of Proposition 3.2 below, it will be convenient to take $q = 5^{2^k}$ for the sequel. Thus, we let \mathbf{F} be the algebraic closure of the field with five elements.

3.1. Automorphisms of the fusion system of $\text{Spin}_7(q)$. Let $\bar{H} = \text{Spin}_7(\mathbf{F})$. Fix a maximal torus \bar{T} of \bar{H} . Thus, \bar{H} is generated by the \bar{T} -root groups $\bar{X}_\alpha = \{x_\alpha(\lambda) : \lambda \in \mathbf{F}\} \cong (\mathbf{F}, +)$, as α ranges over the root system of type B_3 , and is subject to the Chevalley relations of [GLS98, Theorem 1.12.1]. For any power q_1 of 5, we let ψ_{q_1} denote the standard Frobenius endomorphism of \bar{H} , namely the endomorphism of \bar{H} which acts on the root groups via $\psi_{q_1}(x_\alpha(\lambda)) = x_\alpha(\lambda^{q_1})$.

Set $H := C_{\bar{H}}(\psi_q)$. Thus, $H = \text{Spin}_7(q)$ since \bar{H} is of universal type (see [GLS98, Theorem 2.2.6(f)]). Also, $T := C_{\bar{T}}(\psi_q)$ is a maximal torus of H . For each power q_1 of 5, the Frobenius endomorphism ψ_{q_1} of \bar{H} acts on H in the way just mentioned, and it also acts on T by raising each element to the power q_1 . For ease of notation, we denote by ψ_{q_1} also the automorphism of H induced by ψ_{q_1} .

The normalizer $N_H(T)$ contains a Sylow 2-subgroup H [GL83, 10-1(2)], and $N_H(T)/T$ is isomorphic to $C_2 \times S_4$, the Weyl group of B_3 . Applying a Frattini argument, we see that there is a Sylow 2-subgroup S of $N_H(T)$ invariant under ψ_5 , and we fix this choice of S for the remainder.

The automorphism groups of the Chevalley groups were determined by Steinberg [Ste60], and in particular,

$$(3.1) \quad \text{Out}(H) = \text{Outdiag}(H) \times \Phi \cong C_2 \times C_{2^k},$$

where Φ is the group of field automorphisms, and where $\text{Outdiag}(H)$ is the group of outer automorphisms of H induced by $N_{\bar{T}}(H)$ [GLS98, Theorem 2.5.1(b)]. We mention that S is normalized by every element of $N_{\bar{T}}(H)$. So we find canonical representatives of the elements of Φ and of $\text{Outdiag}(H)$ in $\text{Aut}(H, S)$.

We need to be able to compare automorphisms of the group with automorphisms of the fusion and linking systems, and this has been carried out in full generality by Broto, Møller, and Oliver [BMO16] for groups of Lie type.

Let $\mathcal{F}_{\text{Spin}}(q)$ and $\mathcal{L}_{\text{Spin}}^c(q)$ be the associated fusion and centric linking systems over S of the group H , and recall the maps μ_H and κ_H from §§2.2.3, 2.2.4

Proposition 3.2. *The maps μ_H and κ_H are isomorphisms, and hence*

$$\text{Out}(\mathcal{L}_{\text{Spin}}(5^{2^k})) \cong \text{Out}(\mathcal{F}_{\text{Spin}}(5^{2^k})) \cong C_2 \times C_{2^k}.$$

Proof. That μ_H is an isomorphism follows from (2.3) and [LO02, Lemma 3.2]. Also, κ_H is an isomorphism by [BMO16, Propositions 5.14, 5.15]. \square

3.2. Automorphisms of the Benson-Solomon systems. We keep the notation from the previous subsection. We denote by $\mathcal{F} := \mathcal{F}_{\text{Sol}}(q)$ a Benson-Solomon fusion system over the 2-group $S \in \text{Syl}_2(H)$ fixed above, and by $\mathcal{L} := \mathcal{L}_{\text{Sol}}^c(q)$ an associated centric linking system with structural functors δ and π . Set

$$T_2 := T \cap S,$$

the 2-torsion in the maximal torus T of H . Then T_2 is homocyclic of rank three and of exponent 2^{k+2} [AC10, §4]. Also, $Z(S) \leq T_2$ is of order 2, and $N_{\mathcal{F}}(Z(S)) = C_{\mathcal{F}}(Z(S)) = \mathcal{F}_{\text{Spin}}(q)$ is a fusion system over S .

Since $Z(S)$ is contained in every \mathcal{F} -centric subgroup, by Definition 6.1 and Lemma 6.2 of [BLO03], we may take $N_{\mathcal{L}}(Z(S)) = C_{\mathcal{L}}(Z(S))$ for the centric linking system of $\text{Spin}_7(q)$. By the items just referenced, $C_{\mathcal{L}}(Z(S))$ is a subcategory of \mathcal{L} with the same objects, and with morphisms those morphisms φ in \mathcal{L} such that $\pi(\varphi)(z) = z$. Further, $C_{\mathcal{L}}(Z(S))$ has the same inclusion functor δ , and the projection functor for $C_{\mathcal{L}}(Z(S))$ is the restriction of π . (This was also shown in [LO02, Lemma 3.3(a,b)].)

Write \mathcal{F}_z for $\mathcal{F}_{\text{Spin}}(q)$ and \mathcal{L}_z for $C_{\mathcal{L}}(Z(S))$ for short. Each member of $\text{Aut}(\mathcal{F})$ fixes $Z(S)$ and so $\text{Aut}(\mathcal{F}) \subseteq \text{Aut}(\mathcal{F}_z)$. So the inclusion map from $\text{Aut}(\mathcal{F})$ to $\text{Aut}(\mathcal{F}_z)$ can be thought of as a “restriction map”

$$(3.3) \quad \rho: \text{Aut}(\mathcal{F}) \longrightarrow \text{Aut}(\mathcal{F}_z)$$

given by remembering only that an automorphism preserves fusion in \mathcal{F}_z . We want to make explicit in Lemma 3.5 that the map ρ of (3.3) comes from a restriction map on the level of centric linking systems. First we need to recall some information about the normalizer of T_2 in \mathcal{L} and \mathcal{L}_z .

Lemma 3.4. *The following hold after identifying T_2 with its image $\delta_{T_2}(T_2) \leq \text{Aut}_{\mathcal{L}}(T_2)$.*

- (a) $\text{Aut}_{\mathcal{L}_z}(T_2)$ is an extension of T_2 by $C_2 \times S_4$, and $\text{Aut}_{\mathcal{L}}(T_2)$ is an extension of T_2 by $C_2 \times GL_3(2)$ in which the $GL_3(2)$ factor acts naturally on $T_2/\Phi(T_2)$. In each case, a C_2 factor acts as inversion on T_2 . Also, T_2 is equal to its centralizer in each of the above normalizers, $Z(\text{Aut}_{\mathcal{L}_z}(T_2)) = Z(S)$, and $Z(\text{Aut}_{\mathcal{L}}(T_2)) = 1$.
- (b) $\text{Aut}_{\mathcal{F}}(S) = \text{Inn}(S) = \text{Aut}_{\mathcal{F}_z}(S)$ and $\text{Aut}_{\mathcal{L}}(S) = \delta_S(S) = \text{Aut}_{\mathcal{L}_z}(S)$.

Proof. For part (a), see Lemma 4.3 and Proposition 5.4 of [AC10].

Since T_2 is the unique abelian subgroup of its order in S by [AC10, Lemma 4.9(c)], it is characteristic. By the uniqueness of restrictions 2.1.3, we may therefore view $\text{Aut}_{\mathcal{L}}(S)$ as a subgroup of $\text{Aut}_{\mathcal{L}}(T_2)$. Since $\text{Aut}_{\mathcal{L}}(T_2)$ has self-normalizing Sylow 2-subgroups by (a), the same holds for $\text{Aut}_{\mathcal{L}}(S)$. Now (b) follows for \mathcal{L} , and for \mathcal{F} after applying π . This also implies the statement for \mathcal{L}_z and \mathcal{F}_z , as subcategories. \square

There is a 3-dimensional commutative diagram related to (2.3) that is the point of the next lemma.

Lemma 3.5. *There is a restriction map $\hat{\rho}: \text{Aut}(\mathcal{L}) \rightarrow \text{Aut}(\mathcal{L}_z)$, with kernel the automorphisms induced by conjugation by $\delta_S(Z(S)) \leq \text{Aut}_{\mathcal{L}}(S)$, which makes the diagram*

$$\begin{array}{ccc} \text{Aut}(\mathcal{L}) & \xrightarrow{\hat{\rho}} & \text{Aut}(\mathcal{L}_z) \\ \tilde{\mu}_{\mathcal{L}} \downarrow & & \downarrow \tilde{\mu}_{\mathcal{L}_z} \\ \text{Aut}(\mathcal{F}) & \xrightarrow{\rho} & \text{Aut}(\mathcal{F}_z), \end{array}$$

commutative, which commutes with the conjugation maps out of

$$\begin{array}{ccc} \mathrm{Aut}_{\mathcal{L}}(S) & \xrightarrow{\mathrm{id}} & \mathrm{Aut}_{\mathcal{L}_z}(S) \\ \pi_S \downarrow & & \downarrow \pi_S \\ \mathrm{Aut}_{\mathcal{F}}(S) & \xrightarrow{\mathrm{id}} & \mathrm{Aut}_{\mathcal{F}_z}(S), \end{array}$$

and which therefore induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(\mathcal{L}) & \xrightarrow{[\hat{\rho}]} & \mathrm{Out}(\mathcal{L}_z) \\ \mu_{\mathcal{L}} \downarrow & & \downarrow \mu_{\mathcal{L}_z} \\ \mathrm{Out}(\mathcal{F}) & \xrightarrow{[\rho]} & \mathrm{Out}(\mathcal{F}_z). \end{array}$$

Proof. Recall that we have arranged $\mathcal{L}_z \subseteq \mathcal{L}$. Thus, the horizontal maps in the second diagram are the identity maps by Lemma 3.4, and so the lemma amounts to checking that an element of $\mathrm{Aut}(\mathcal{L})$ sends morphisms in \mathcal{L}_z to morphisms in \mathcal{L}_z . For then, we can define its image under $\hat{\rho}$ to have the same effect on objects, and to be the restriction to \mathcal{L}_z on morphisms.

Now fix an arbitrary $\alpha \in \mathrm{Aut}(\mathcal{L})$, objects $P, Q \in \mathcal{F}^c = \mathcal{F}_z^c$, and a morphism $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$. Let $Z(S) = \langle z \rangle$. By two applications of Axiom (C) for a linking system (§§2.1.2),

$$(3.6) \quad \iota_{P,S} \circ \delta_P(z) = \delta_S(z) \circ \iota_{P,S} \quad \text{and} \quad \iota_{\alpha(P),S} \circ \delta_{\alpha(P)}(z) = \delta_S(z) \circ \iota_{\alpha(P),S},$$

because $\pi(\iota_{P,S})(z) = \pi(\iota_{\alpha(P),S})(z) = z$. Since α_S is an automorphism of $\delta_S(S) \cong S$, it sends $\delta_S(z)$ to itself. Thus, α sends the right side of the first equation of (3.6) to the right side of the second, since it sends inclusions to inclusions. Thus

$$\iota_{\alpha(P),S} \circ \alpha(\delta_P(z)) = \iota_{\alpha(P),S} \circ \delta_{\alpha(P)}(z).$$

However, each morphism in \mathcal{L} is a monomorphism [Oli10, Proposition 4], so we obtain

$$(3.7) \quad \alpha(\delta_P(z)) = \delta_{\alpha(P)}(z),$$

and the same holds for Q in place of P .

Since $\varphi \in \mathrm{Mor}(\mathcal{L}_z)$, we have $\pi(\varphi)(z) = z$, so by two more applications of Axiom (C),

$$(3.8) \quad \varphi \circ \delta_P(z) = \delta_Q(z) \circ \varphi \quad \text{and} \quad \alpha(\varphi) \circ \delta_{\alpha(P)}(z) = \delta_{\alpha(Q)}(\pi(\alpha(\varphi))(z)) \circ \alpha(\varphi).$$

After applying α to the left side of the first equation of (3.8), we obtain the left side of the second by (3.7). Thus, comparing right sides, we obtain

$$\delta_{\alpha(Q)}(z) \circ \alpha(\varphi) = \delta_{\alpha(Q)}(\pi(\alpha(\varphi))(z)) \circ \alpha(\varphi)$$

Since each morphism in \mathcal{L} is an epimorphism [Oli10, Proposition 4], it follows that

$$\delta_{\alpha(Q)}(z) = \delta_{\alpha(Q)}(\pi(\alpha(\varphi))(z)).$$

Hence, $\pi(\alpha(\varphi))(z) = z$ because $\delta_{\alpha(Q)}$ is injective (Axiom (A2)). That is, $\alpha(\varphi) \in \mathrm{Mor}(\mathcal{L}_z)$ as required.

The kernel of $\hat{\rho}$ is described via a diagram chase in (2.3). Suppose $\hat{\rho}(\alpha)$ is the identity. Then, α is sent to the identity automorphism of S by $\tilde{\mu}_{\mathcal{L}}$, since ρ is injective. Thus, α comes from a normalized 1-cocycle by (2.3) and these are in turn induced by elements of $Z(S)$ since $\varprojlim^1(\mathcal{Z}_{\mathcal{F}})$ is trivial [LO02, Lemma 3.2]. \square

Lemma 3.9. *Let G be a finite group and let V be an abelian normal 2-subgroup of G such that $C_G(V) \leq V$. Let α be an automorphism of G such that $[V, \alpha] = 1$ and $\alpha^2 \in \text{Inn}(G)$. Then $[G, \alpha] \leq V$, and if G acts fixed point freely on $V/\Phi(V)$, then the order of α is at most the exponent of V .*

Proof. As $[V, \alpha] = 1$, we have $[V, G, \alpha] \leq [V, \alpha] = 1$ and $[\alpha, V, G] = [1, G] = 1$. Hence, by the Three subgroups lemma, it follows $[G, \alpha, V] = 1$. As $C_G(V) \leq V$, this means

$$[G, \alpha] \leq V.$$

Assume from now on that G acts fixed point freely on $V/\Phi(V)$. Write $G^* := G \rtimes \langle \alpha \rangle$ for the semidirect product of G by $\langle \alpha \rangle$. As $[V, \alpha] = 1$ and $[G, \alpha] \leq V$, the subgroup $W := V\langle \alpha \rangle$ is an abelian normal subgroup of G^* with $[W, G^*] \leq V$.

As $[V, \alpha] = 1$, it follows $[V, \alpha^2] = 1$. So $\alpha^2 \in \text{Inn}(G)$ is realized by conjugation with an element of $C_G(V) = V$. Pick $u \in V$ with $\alpha^2 = c_u|_G$. This means that, for any $g \in G$, we have $u^{-1}\alpha^2 g = g$ in G^* . So $Z := \langle u^{-1}\alpha^2 \rangle$ centralizes G in G^* . Since W is abelian and contains Z , it follows that Z lies in the centre of $G^* = WG$. Set

$$\overline{G^*} = G^*/Z.$$

Because $C_G(V) \leq V$, the order of u equals the order of $c_u|_G = \alpha^2$. Hence, $Z \cap G = 1 = Z \cap \langle \alpha \rangle$. So $|\bar{\alpha}| = |\alpha|$ and $G \cong \bar{G}$. In particular, we have $\bar{V} \cong V$ and \bar{G} acts fixed point freely on $\bar{V}/\Phi(\bar{V})$. Note also that $\bar{\alpha}^2 = \bar{u}$. Hence $|\bar{W}/\bar{V}| = 2$ and $\Phi(\bar{W}) \leq \bar{V}$. Moreover, letting $n \in \mathbb{N}$ such that 2^n is the exponent of V , we have $|\alpha| = |\bar{\alpha}| \leq 2 \cdot 2^n = 2^{n+1}$. Assume $|\bar{\alpha}| = 2^{n+1}$. Then $\bar{u} = \bar{\alpha}^2$ has order 2^n and is thus not a square in \bar{V} . Note that $\Phi(\bar{V}) = \{v^2 : v \in \bar{V}\}$ and $\Phi(\bar{W}) = \{w^2 : w \in \bar{W}\} = \langle \bar{\alpha}^2 \rangle \Phi(\bar{V}) \leq \bar{V}$. Hence, $\Phi(\bar{W})/\Phi(\bar{V})$ has order 2. As \bar{G} normalizes $\Phi(\bar{W})/\Phi(\bar{V})$, it thus centralizes $\Phi(\bar{W})/\Phi(\bar{V})$ contradicting the assumption that \bar{G} acts fixed point freely on $\bar{V}/\Phi(\bar{V})$. Thus $|\alpha| = |\bar{\alpha}| \leq 2^n$ which shows the assertion. \square

We are now in a position to determine the automorphisms of $\mathcal{F} = \mathcal{F}_{\text{Sol}(q)}$ and $\mathcal{L} = \mathcal{L}_{\text{Sol}(q)}^c$. It is known that the field automorphisms induce automorphisms of these systems as we will make precise next. Recall that the field automorphism ψ_5 of H of order 2^k normalizes S and so $\psi_5|_S$ is an automorphism of $\mathcal{F}_z = \mathcal{F}_S(H)$. By [AC10, Lemma 5.7], the automorphism $\psi_5|_S$ is actually also an automorphism of \mathcal{F} . We thus denote it by $\psi_{\mathcal{F}}$ and refer to it as the field automorphism of \mathcal{F} induced by ψ_5 . By Proposition 3.2, this automorphism has order 2^k .

By [LO02, Proposition 3.3(d)], there is a unique lift ψ of $\psi_{\mathcal{F}}$ under $\tilde{\mu}_{\mathcal{L}}$ that is the identity on $\pi^{-1}(\mathcal{F}_{\text{Sol}(5)})$ and restricts to $\tilde{\kappa}_H(\psi_5)$ on \mathcal{L}_z . We refer to ψ as the field automorphism of \mathcal{L} induced by ψ_5 .

Theorem 3.10. *The map $\mu_{\mathcal{L}} : \text{Out}(\mathcal{L}_{\text{Sol}(q)}^c) \rightarrow \text{Out}(\mathcal{F}_{\text{Sol}(q)})$ is an isomorphism, and*

$$\text{Out}(\mathcal{L}_{\text{Sol}(q)}^c) \cong \text{Out}(\mathcal{F}_{\text{Sol}(q)}) \cong C_{2^k}$$

is induced by field automorphisms. Also, the automorphism group $\text{Aut}(\mathcal{L}_{\text{Sol}(q)}^c)$ is a split extension of S by $\text{Out}(\mathcal{L}_{\text{Sol}(q)}^c)$; in particular, it is a 2-group.

More precisely, if ψ is the field automorphism of $\mathcal{L}_{\text{Sol}(q)}^c$ induced by ψ_5 , then ψ has order 2^k and $\text{Aut}(\mathcal{L}_{\text{Sol}(q)}^c)$ is the semidirect product of $\text{Aut}_{\mathcal{L}}(S) \cong S$ with the cyclic group generated by ψ .

Proof. We continue to write $\mathcal{L} = \mathcal{L}_{\text{Sol}(q)}^c$, $\mathcal{F} = \mathcal{F}_{\text{Sol}(q)}$, $\mathcal{L}_z = \mathcal{L}_{\text{Spin}(q)}^c$, and $\mathcal{F}_z = \mathcal{F}_{\text{Spin}(q)}$, and we continue to assume that \mathcal{L} has been chosen so as to contain \mathcal{L}_z as a linking subsystem. Recall that $T_2 \leq S$ is homocyclic of rank 3 and exponent 2^{k+2} .

We first check whether the outer automorphism of \mathcal{L}_z induced by a diagonal automorphism of H extends to \mathcal{L} , and we claim that it doesn't. A non-inner diagonal automorphism of H

is induced by conjugation by an element t of \bar{T} by [GLS98, Theorem 2.5.1(b)]. Its class as an outer automorphism has order 2, so if necessary we replace t by an odd power and assume that $t^2 \in T_2$. Now T_2 consists of the elements of \bar{T} of order at most 2^{k+2} , so t has order 2^{k+3} and induces an automorphism of H of order at least 2^{k+2} , depending on whether it powers into $Z(H) = Z(S)$ or not (in fact it does, but this is not needed). For ease of notation, we identify T_2 with $\delta_{T_2}(T_2) \leq \text{Aut}_{\mathcal{L}}(T_2)$, and we identify $s \in S$ with $\delta_S(s) \in \text{Aut}_{\mathcal{L}}(S)$.

Let $\tau = \tilde{\kappa}_H([c_t]) \in \text{Aut}(\mathcal{L}_z)$, and assume that τ lifts to an element $\hat{\tau} \in \text{Aut}(\mathcal{L})$ under the map $\hat{\rho}$ of Lemma 3.5. As $\hat{\rho}(\hat{\tau}) = \tau = \tilde{\kappa}_H(c_t)$, we have $\hat{\rho}(\hat{\tau}^2) = \tau^2 = \tilde{\kappa}_H(c_{t^2})$, i.e. $\hat{\rho}(\hat{\tau}^2)$ acts on every object and every morphism of $\mathcal{L}_z = \mathcal{L}_S^c(H)$ as conjugation by t^2 . Similarly, if we take the conjugation automorphism c_{t^2} of \mathcal{L} by t^2 (or more precisely the conjugation automorphism $c_{\delta_S(t^2)}$ of \mathcal{L} by $\delta_S(t^2)$), then $\hat{\rho}(c_{t^2})$ is just the conjugation automorphism of \mathcal{L}_z by t^2 . So according to the remark at the end of §2.2.2, the automorphism $\hat{\rho}(c_{t^2})$ acts also on \mathcal{L}_z via conjugation by t^2 , showing $\hat{\rho}(\hat{\tau}^2) = \hat{\rho}(c_{t^2})$. By the description of the kernel in Lemma 3.5, we have thus $\hat{\tau}^2 = c_{t^2}$ or $\hat{\tau}^2 = c_{t^2 z}$.

Now set $\alpha := \hat{\tau}_{T_2} \in \text{Aut}(\text{Aut}_{\mathcal{L}}(T_2))$. From what we have shown, it follows that α equals the conjugation automorphism c_{t^2} or $c_{t^2 z}$ of $\text{Aut}_{\mathcal{L}}(T_2)$. Note that $|c_{t^2 z}| = |t^2 z| = |t^2| = |c_{t^2}|$, since $Z(\text{Aut}_{\mathcal{L}}(T_2)) = 1$ by Lemma 3.4(a). Hence,

$$(3.11) \quad |\alpha| = 2|\alpha^2| = 2|t^2| = 2^{k+3},$$

On the other hand, α centralizes T_2 , and we have seen that α^2 is an inner automorphism of $\text{Aut}_{\mathcal{L}}(T_2)$. Moreover, by Lemma 3.4(a), $C_{\text{Aut}_{\mathcal{L}}(T_2)}(T_2) = T_2$, and $\text{Aut}_{\mathcal{L}}(T_2)$ acts fixed point freely on $T_2/\Phi(T_2)$. The hypotheses of Lemma 3.9 thus hold for $G = \text{Aut}_{\mathcal{L}}(T_2)$ and $\alpha \in \text{Aut}(G)$. So α has order at most 2^{k+2} by that lemma, contradicting (3.11). We conclude that a diagonal automorphism of \mathcal{L}_z does not extend to an automorphism of \mathcal{L} .

The existence of the field automorphism $\psi_{\mathcal{F}}$ of \mathcal{F} , and the fact that $\psi_{\mathcal{F}}$ has order 2^k , now yields together with Proposition 3.2 that $\text{Out}(\mathcal{F}) \cong C_{2^k}$ is generated by the image of $\psi_{\mathcal{F}}$ in $\text{Out}(\mathcal{F})$. Moreover, by [LO02, Lemma 3.2] and the exactness of the third column of (2.3), the maps $\mu_{\mathcal{L}}$ and $\mu_{\mathcal{L}_z}$ are isomorphisms. Thus,

$$\text{Out}(\mathcal{L}) \cong \text{Out}(\mathcal{F}) \cong C_{2^k}.$$

Let ψ be the field automorphism of \mathcal{L} induced by ψ_5 as above. Then ψ is the identity on $\pi^{-1}(\mathcal{F}_{\text{Sol}}(5))$ by definition. It remains to show that ψ has order 2^k , since this will imply that $\text{Aut}(\mathcal{L})$ is a split extension of $\text{Aut}_{\mathcal{L}}(S) \cong S$ by $\langle \psi \rangle \cong \text{Out}(\mathcal{L}) \cong \text{Out}(\mathcal{F})$.

The automorphism ψ^{2^k} maps to the trivial automorphism of \mathcal{F} , and so is conjugation by an element of $Z(S)$ by (2.3). Now ψ^{2^k} is trivial on $\text{Aut}_{\mathcal{L}_{\text{Sol}}^c(5)}(\Omega_2(T_2))$, whereas $z \notin Z(\text{Aut}_{\mathcal{L}_{\text{Sol}}^c(5)}(\Omega_2(T_2)))$ by Lemma 3.4(a) as $\Omega_2(T_2)$ is the torus of $\mathcal{L}_{\text{Sol}}^c(5)$. Thus, since a morphism φ is fixed by c_z if and only if $\pi(\varphi)(z) = z$ (Axiom (C)), we conclude that ψ^{2^k} is the identity automorphism of \mathcal{L} , and this completes the proof. \square

4. EXTENSIONS

In this section, we recall a result of Linckelmann on the Schur multipliers of the Benson-Solomon systems, and we prove that each saturated fusion system \mathcal{F} with $F^*(\mathcal{F}) \cong \mathcal{F}_{\text{Sol}}(q)$ is a split extension of $F^*(\mathcal{F})$ by a group of outer automorphisms.

Recall that the *hyperfocal subgroup* of a saturated p -fusion system \mathcal{F} over S is defined to be the subgroup of S given by

$$\text{hyp}(\mathcal{F}) = \langle [\varphi, s] := \varphi(s)s^{-1} \mid s \in P \leq S \text{ and } \varphi \in O^p(\text{Aut}_{\mathcal{F}}(P)) \rangle.$$

A subsystem \mathcal{F}_0 over $S_0 \leq S$ is said to be of *p-power index* in \mathcal{F} if $\mathfrak{hnp}(\mathcal{F}) \leq S_0$ and $Op(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$ for each $P \leq S_0$. There is always a unique normal saturated subsystem on $\mathfrak{hnp}(\mathcal{F})$ of *p-power index* in \mathcal{F} , which is denoted by $Op(\mathcal{F})$ [AKO11, §I.7]. We will need the next lemma in §§4.2.

Lemma 4.1. *Let \mathcal{F} be a saturated fusion system over S , and let \mathcal{F}_0 be a weakly normal subsystem of \mathcal{F} over $S_0 \leq S$. Assume that $Op(\text{Aut}_{\mathcal{F}}(S_0)) \leq \text{Aut}_{\mathcal{F}_0}(S_0)$. Then $Op(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$ for every $P \leq S_0$. Thus, if in addition $\mathfrak{hnp}(\mathcal{F}) \leq S_0$, then \mathcal{F}_0 has *p-power index* in \mathcal{F} .*

Proof. Note that $\text{Aut}_{\mathcal{F}_0}(P)$ is normal in $\text{Aut}_{\mathcal{F}}(P)$ for every $P \leq S_0$, since \mathcal{F}_0 is weakly normal in \mathcal{F} . We need to show that $\text{Aut}_{\mathcal{F}}(P)/\text{Aut}_{\mathcal{F}_0}(P)$ is a *p*-group for every $P \leq S_0$. Suppose this is false and let P be a counterexample of maximal order. Our assumption gives $P < S_0$. Hence, $P < Q := N_{S_0}(P)$, and the maximality of P implies that

$$\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_{\mathcal{F}_0}(Q)$$

is a *p*-group. Notice that

$$\begin{aligned} N_{\text{Aut}_{\mathcal{F}}(Q)}(P)/N_{\text{Aut}_{\mathcal{F}_0}(Q)}(P) &\cong N_{\text{Aut}_{\mathcal{F}}(Q)}(P)\text{Aut}_{\mathcal{F}_0}(Q)/\text{Aut}_{\mathcal{F}_0}(Q) \\ &\leq \text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_{\mathcal{F}_0}(Q) \end{aligned}$$

and thus $N_{\text{Aut}_{\mathcal{F}}(Q)}(P)/N_{\text{Aut}_{\mathcal{F}_0}(Q)}(P)$ is a *p*-group.

If $\alpha \in \text{Hom}_{\mathcal{F}}(P, S)$ with $\alpha(P) \in \mathcal{F}^f$ then conjugation by α induces a group isomorphism from $\text{Aut}_{\mathcal{F}}(P)$ to $\text{Aut}_{\mathcal{F}}(\alpha(P))$. As \mathcal{F}_0 is weakly normal, we have $\alpha(P) \leq S_0$ and conjugation by α takes $\text{Aut}_{\mathcal{F}_0}(P)$ to $\text{Aut}_{\mathcal{F}_0}(\alpha(P))$. So upon replacing P by $\alpha(P)$, we may assume without loss of generality that P is fully \mathcal{F} -normalized. Then P is also fully \mathcal{F}_0 -normalized by [Asc08, Lemma 3.4(5)]. By the Sylow axiom, $\text{Aut}_{S_0}(P)$ is a Sylow *p*-subgroup of $\text{Aut}_{\mathcal{F}_0}(P)$. So the Frattini argument yields

$$\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}_0}(P)N_{\text{Aut}_{\mathcal{F}}(P)}(\text{Aut}_{S_0}(P))$$

and thus

$$\text{Aut}_{\mathcal{F}}(P)/\text{Aut}_{\mathcal{F}_0}(P) \cong N_{\text{Aut}_{\mathcal{F}}(P)}(\text{Aut}_{S_0}(P))/N_{\text{Aut}_{\mathcal{F}_0}(P)}(\text{Aut}_{S_0}(P)).$$

By the extension axiom for \mathcal{F} and \mathcal{F}_0 , each element of $N_{\text{Aut}_{\mathcal{F}}(P)}(\text{Aut}_{S_0}(P))$ extends to an automorphism of $\text{Aut}_{\mathcal{F}}(Q)$, and each element of $N_{\text{Aut}_{\mathcal{F}_0}(P)}(\text{Aut}_{S_0}(P))$ extends to an automorphism of $\text{Aut}_{\mathcal{F}_0}(Q)$. Therefore, the map

$$\Phi: N_{\text{Aut}_{\mathcal{F}}(Q)}(P) \rightarrow N_{\text{Aut}_{\mathcal{F}}(P)}(\text{Aut}_{S_0}(P)), \varphi \mapsto \varphi|_P$$

is an epimorphism which maps $N_{\text{Aut}_{\mathcal{F}_0}(Q)}(P)$ onto $N_{\text{Aut}_{\mathcal{F}_0}(P)}(\text{Aut}_{S_0}(P))$. Hence,

$$\begin{aligned} \text{Aut}_{\mathcal{F}}(P)/\text{Aut}_{\mathcal{F}_0}(P) &\cong N_{\text{Aut}_{\mathcal{F}}(P)}(\text{Aut}_{S_0}(P))/N_{\text{Aut}_{\mathcal{F}_0}(P)}(\text{Aut}_{S_0}(P)) \\ &\cong N_{\text{Aut}_{\mathcal{F}}(Q)}(P)/N_{\text{Aut}_{\mathcal{F}_0}(Q)}(P) \ker(\Phi). \end{aligned}$$

We have seen above that $N_{\text{Aut}_{\mathcal{F}}(Q)}(P)/N_{\text{Aut}_{\mathcal{F}_0}(Q)}(P)$ is a *p*-group, and therefore also

$$N_{\text{Aut}_{\mathcal{F}}(Q)}(P)/N_{\text{Aut}_{\mathcal{F}_0}(Q)}(P) \ker(\Phi)$$

is a *p*-group. Hence, $\text{Aut}_{\mathcal{F}}(P)/\text{Aut}_{\mathcal{F}_0}(P)$ is a *p*-group, and this contradicts our assumption that P is a counterexample. \square

4.1. Extensions to the bottom. A *central extension* of a fusion system \mathcal{F}_0 is a fusion system \mathcal{F} such that $\mathcal{F}/Z \cong \mathcal{F}_0$ for some subgroup $Z \leq Z(\mathcal{F})$. The central extension is said to be *perfect* if $\mathcal{F} = O^p(\mathcal{F})$. Linckelmann has shown that the Schur multiplier of a Benson-Solomon system is trivial.

Theorem 4.2 (Linckelmann). *Let \mathcal{F} be a perfect central extension of a Benson-Solomon fusion system \mathcal{F}_0 . Then $\mathcal{F} = \mathcal{F}_0$.*

Proof. This follows from Corollary 4.4 of [Lin06a] together with the fact that $\text{Spin}_7(q)$ has Schur multiplier of odd order when q is odd [GLS98, Tables 6.1.2, 6.1.3]. \square

4.2. Extensions to the top. The next theorem describes the possible extensions (S, \mathcal{F}) of a Benson-Solomon system (S_0, \mathcal{F}_0) . The particular hypotheses are best stated in terms of the generalized Fitting subsystem of Aschbacher [Asc11], but they are equivalent to requiring that $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ and $C_S(\mathcal{F}_0) \leq S_0$, where $C_S(\mathcal{F}_0)$ is the centralizer constructed in [Asc11, §6]. This latter formulation is sometimes expressed by saying that \mathcal{F}_0 is *centric normal* in \mathcal{F} .

Theorem 4.3. *Let $\mathcal{F}_0 = \mathcal{F}_{\text{Sol}}(5^{2^k})$ be a Benson-Solomon system over S_0 .*

- (a) *If \mathcal{F} is a saturated fusion system over S such that $F^*(\mathcal{F}) = \mathcal{F}_0$, then $\mathcal{F}_0 = O^2(\mathcal{F})$, S splits over S_0 , and the map $S/S_0 \rightarrow \text{Out}(\mathcal{F}_0)$ induced by conjugation is injective.*
- (b) *Conversely, given a subgroup of $A \leq \text{Out}(\mathcal{F}_0) \cong C_{2^k}$, there is a saturated fusion system \mathcal{F} over some 2-group S such that $F^*(\mathcal{F}) = \mathcal{F}_0$ and the map $S/S_0 \rightarrow \text{Out}(\mathcal{F}_0)$ induced by conjugation on S_0 has image A . Moreover, the pair (S, \mathcal{F}) with these properties is uniquely determined up to isomorphism.*

If \mathcal{L}_0 is a centric linking system associated to \mathcal{F}_0 , then $\text{Aut}_{\mathcal{L}_0}(S_0) = S_0$, and the p -group S can be chosen to be the preimage of A in $\text{Aut}(\mathcal{L}_0)$ under the quotient map from $\text{Aut}(\mathcal{L}_0)$ to $\text{Out}(\mathcal{L}_0) \cong \text{Out}(\mathcal{F}_0)$.

Proof. Let \mathcal{F} be a saturated fusion system over S such that $F^*(\mathcal{F}) = \mathcal{F}_0$. Set $\mathcal{F}_1 = \mathcal{F}_0 S$, the internal extension of \mathcal{F}_0 by S , as in [Hen13] or [Asc11, §8]. According to [AOV12, Proposition 1.31], there is a normal pair of linking systems $\mathcal{L}_0 \trianglelefteq \mathcal{L}_1$, associated to the normal pair $\mathcal{F}_0 \trianglelefteq \mathcal{F}_1$. Furthermore, $\mathcal{L}_0 \trianglelefteq \mathcal{L}_1$ can be chosen such that \mathcal{L}_0 is a centric linking system. There is a natural map from $\text{Aut}_{\mathcal{L}_1}(S_0)$ to $\text{Aut}(\mathcal{L}_0)$ which sends a morphism $\varphi \in \text{Aut}_{\mathcal{L}_1}(S_0)$ to conjugation by φ . (So the restriction of this map to $\text{Aut}_{\mathcal{L}_0}(S_0)$ is the conjugation map described in §§2.2.2.)

The centralizer $C_S(\mathcal{F}_0)$ depends a priori on the fusion system \mathcal{F} , but it is shown in [Lyn15, Lemma 1.13] that it does not actually matter whether we form $C_S(\mathcal{F}_0)$ inside of \mathcal{F} or inside of \mathcal{F}_1 . Moreover, since $F^*(\mathcal{F}) = \mathcal{F}_0$, it follows from [Asc11, Theorem 6] that $C_S(\mathcal{F}_0) = Z(\mathcal{F}_0) = 1$. Thus, by a result of Semeraro [Sem15, Theorem A], the conjugation map $\text{Aut}_{\mathcal{L}_1}(S_0) \xrightarrow{\text{conj}} \text{Aut}(\mathcal{L}_0)$ is injective. By Lemma 3.4, we have $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ via the inclusion functor δ_1 for \mathcal{L}_1 . By Theorem 3.10, $\text{Aut}(\mathcal{L}_0)$ is a 2-group which splits over S_0 . Moreover, by the same theorem, we have that $C_{\text{Aut}(\mathcal{L}_0)}(S_0) \leq S_0$ and $\text{Out}(\mathcal{L}_0) \cong \text{Out}(\mathcal{F}_0)$ is cyclic. Since $(\delta_1)_{S_0}(S) \cong S$ is a Sylow 2-subgroup of $\text{Aut}_{\mathcal{L}_1}(S_0)$ by [Oli10, Proposition 4(d)], we can conclude that

$$S_0 = \text{Aut}_{\mathcal{L}_0}(S_0) \trianglelefteq \text{Aut}_{\mathcal{L}_1}(S_0) = S,$$

via the inclusion functor δ_1 for \mathcal{L}_1 . Moreover, it follows that S splits over S_0 , and $C_S(S_0) \leq S_0$. The latter property means that the map

$$S/S_0 \rightarrow \text{Out}(\mathcal{F}_0)$$

is injective. In particular, S/S_0 is cyclic as $\text{Out}(\mathcal{F}_0)$ is cyclic.

Next, we show that $O^2(\mathcal{F}) = \mathcal{F}_0$. Fix a subgroup $P \leq S$, and let $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ be an automorphism of odd order. Then α induces an odd-order automorphism of the cyclic 2-group $P/(P \cap S_0) \cong PS_0/S_0 \leq S/S_0$. This automorphism must be trivial, and so $[P, \alpha] \leq S_0$. Hence, $[P, O^2(\text{Aut}_{\mathcal{F}}(P))] \leq S_0$ for all $P \leq S$. Since $\text{hnp}(\mathcal{F}_0) = S_0$, we have $\text{hnp}(\mathcal{F}) = S_0$. Note that $\text{Aut}_{\mathcal{F}}(S_0)$ is a 2-group as $\text{Aut}_{\mathcal{F}}(S_0) \leq \text{Aut}(\mathcal{F}_0)$ and $\text{Aut}(\mathcal{F}_0)$ is a 2-group by Theorem 3.10. Therefore $O^2(\mathcal{F}) = \mathcal{F}_0$ by Lemma 4.1. We conclude that $\mathcal{F}_1 = \mathcal{F}$ by the uniqueness statement in [Hen13, Theorem 1]. This completes the proof of (a). Moreover, we have seen that the following property holds for any normal pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ attached to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$:

$$(4.4) \quad S_0 = \text{Aut}_{\mathcal{L}_0}(S_0) \trianglelefteq \text{Aut}_{\mathcal{L}_1}(S_0) = S \text{ and } S \xrightarrow{\text{conj}} \text{Aut}(\mathcal{L}_0) \text{ is injective.}$$

Finally, we prove (b). Fix a centric linking system \mathcal{L}_0 associated to \mathcal{F}_0 with inclusion functor δ_0 . Let $S \leq \text{Aut}(\mathcal{L}_0)$ be the preimage of A under the quotient map to $\text{Out}(\mathcal{F}_0)$. We will identify S_0 with $\delta_0(S_0)$ so that $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ by Lemma 3.4. Write $\iota: S_0 \rightarrow \text{Aut}(\mathcal{L}_0), s \mapsto c_s$ for map sending $s \in S_0$ to the automorphism of \mathcal{L}_0 induced by conjugation with s in \mathcal{L}_0 . Then $\iota(S_0)$ is normal in S . Let $\chi: S \rightarrow \text{Aut}(S_0)$ be the map defined by $\alpha \mapsto \iota^{-1} \circ c_{\alpha}|_{\iota(S_0)} \circ \iota$; i.e. χ corresponds to conjugation in S if we identify S_0 with $\iota(S_0)$. We argue next that the following diagram commutes:

$$(4.5) \quad \begin{array}{ccc} S_0 & \xrightarrow{\iota} & \text{Aut}(\mathcal{L}_0) \\ \downarrow \iota & \nearrow \text{incl} & \downarrow \alpha \mapsto \alpha_{S_0} \\ S & \xrightarrow{\chi} & \text{Aut}(S_0) \end{array}$$

The upper triangle clearly commutes. Observe that $\alpha \circ \iota(s) \circ \alpha^{-1} = \alpha \circ c_s \circ \alpha^{-1} = c_{\alpha_{S_0}(s)} = \iota(\alpha_{S_0}(s))$ for every $s \in S_0$ and $\alpha \in S$. Hence, for every $\alpha \in S$ and $s \in S_0$, we have $(\iota^{-1} \circ c_{\alpha}|_{\iota(S_0)} \circ \iota)(s) = \iota^{-1}(\alpha \circ \iota(s) \circ \alpha^{-1}) = \alpha_{S_0}(s)$ and so the lower triangle commutes.

We will now identify S_0 with its image in S under ι , so that ι becomes the inclusion map and χ corresponds to the map $S \rightarrow \text{Aut}(S_0)$ induced by conjugation in S . As the above diagram commutes, it follows then that the diagram in [Oli10, Theorem 9] commutes when we take $\Gamma = S$ and $\tau: S \rightarrow \text{Aut}(\mathcal{L}_0)$ to be the inclusion. Thus, by that theorem, there is a saturated fusion system \mathcal{F} over S in which \mathcal{F}_0 is weakly normal, and there is a corresponding normal pair of linking systems $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ (in the sense of [AOV12, §1.5]) such that $S = \text{Aut}_{\mathcal{L}}(S_0)$ has the given action on \mathcal{L}_0 (i.e. the automorphism of \mathcal{L}_0 induced by conjugation with $s \in S$ in \mathcal{L} equals the automorphism s of \mathcal{L}_0). By the same theorem, the pair $(\mathcal{F}, \mathcal{L})$ is unique up to isomorphism of fusion systems and linking systems with these properties. Since \mathcal{F}_0 is simple [Lin06b], \mathcal{F}_0 is in fact normal in \mathcal{F} by a result of Craven [Cra11, Theorem A]. Thus, since $C_S(\mathcal{F}_0) \leq C_S(S_0) \leq S_0$, it is a consequence of [Asc11, (9.1)(2), (9.6)] that $F^*(\mathcal{F}) = \mathcal{F}_0$.

So it remains only to prove that (S, \mathcal{F}) is uniquely determined up to an isomorphism of fusion systems. Let \mathcal{F}' be a saturated fusion system over a p -group S' such that $F^*(\mathcal{F}') = \mathcal{F}_0$, and such that the map $S'/S_0 \rightarrow \text{Out}(\mathcal{F}_0)$ induced by conjugation has image A . Then by (a), $\mathcal{F}_0 = O^p(\mathcal{F}')$. So by [AOV12, Proposition 1.31], there is a normal pair of linking systems $\mathcal{L}'_0 \trianglelefteq \mathcal{L}'$ associated to the normal pair $\mathcal{F}_0 \trianglelefteq \mathcal{F}'$. Moreover, we can choose \mathcal{L}'_0 to be a centric linking system. Since a centric linking system attached to \mathcal{F}_0 is unique, there is an isomorphism $\theta: \mathcal{L}'_0 \rightarrow \mathcal{L}_0$ of linking systems. We may assume that the set of morphisms which lie in \mathcal{L}' but not in \mathcal{L}'_0 is disjoint from the set of morphisms in \mathcal{L}_0 . Then we can construct a new linking system from \mathcal{L}' by keeping every morphism of \mathcal{L}' which is not in \mathcal{L}'_0 and replacing every morphism ψ in \mathcal{L}'_0 by $\theta(\psi)$, and then carrying over the structure of \mathcal{L}' in the natural way. Thereby we may assume $\mathcal{L}'_0 = \mathcal{L}_0$. So we are

given a normal pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}'$ attached to $\mathcal{F}_0 \trianglelefteq \mathcal{F}'$. By (4.4) applied with \mathcal{L}' and \mathcal{F}' in place of \mathcal{F} and \mathcal{L} , we have $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0) \trianglelefteq \text{Aut}_{\mathcal{L}'}(S_0) = S'$ via the inclusion functor δ' of \mathcal{L}' . Let

$$\tau: S' \rightarrow \text{Aut}(\mathcal{L}_0)$$

be the map taking $s \in S'$ to the automorphism of \mathcal{L}_0 induced by conjugation with s in \mathcal{L}' . Again using (4.4), we see that τ is injective. Note also that τ restricts to the identity on S_0 if we identify S_0 with $\iota(S_0)$ as above. Recall that the map $S'/S_0 \rightarrow \text{Out}(\mathcal{F}_0)$ induced by conjugation has image A . So Theorem 3.10 implies $\tau(S') = S$, i.e. we can regard τ as an isomorphism $\tau: S' \rightarrow S$. So replacing (S', \mathcal{F}') by (S, \mathcal{F}) and then choosing $\mathcal{L}_0 \trianglelefteq \mathcal{L}'$ as before, we may assume $S = S'$. So \mathcal{F}' is a fusion system over S with $\mathcal{F}_0 \trianglelefteq \mathcal{F}'$, and $\mathcal{L}_0 \trianglelefteq \mathcal{L}'$ is a normal pair of linking systems associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}'$ such that $\text{Aut}_{\mathcal{L}'}(S_0) = S$ via δ' . Let $s \in S$. Recall that $\tau(s)$ is the automorphism of \mathcal{L}_0 induced by conjugation with s in \mathcal{L}' . Observe that the automorphism of $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ induced by $\tau(s)$ equals just the automorphism of S_0 induced by conjugation with s in S . Similarly, the automorphism s of \mathcal{L}_0 equals the automorphism of \mathcal{L}_0 given by conjugation with s in \mathcal{L} , and so induces on $S_0 = \text{Aut}_{\mathcal{L}_0}(S_0)$ just the automorphism given by conjugation with s in S . Theorem 3.10 gives $C_{\text{Aut}(\mathcal{L}_0)}(S_0) \leq S_0$ and this implies that any two automorphisms of \mathcal{L}_0 , which induce the same automorphism on S_0 , are equal. Hence, $\tau(s) = s$ for any $s \in S$. In other words, $S = \text{Aut}_{\mathcal{L}'}(S_0)$ induces by conjugation in \mathcal{L}' the canonical action of S on \mathcal{L}_0 . The uniqueness of the pair $(\mathcal{F}, \mathcal{L})$ implies now $\mathcal{F}' \cong \mathcal{F}$ and $\mathcal{L}' \cong \mathcal{L}$. This shows that (S, \mathcal{F}) is uniquely determined up to isomorphism. \square

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